

## SEMI-SYMMETRIC GENERALIZED SASAKIAN-SPACE-FORMS

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**ABSTRACT.** We show that semi symmetric and pseudo symmetric generalized Sasakian-space-forms are Einstein when  $(0, 6)$ -tensors satisfy  $R \cdot R = 0$ ,  $R \cdot R = L_R Q(g, R)$ ,  $R \cdot C = 0$ ,  $R \cdot C = L_C Q(g, C)$ , and  $C \cdot C = 0$ , where  $C$  is Quasi conformal curvature tensor. Further we discuss about Ricci solitons.

### 1. INTRODUCTION

Let  $(M, g)$  be a  $(2n + 1)$  dimensional Riemannian manifold and let  $\nabla$  be its Levi-Civita connection. The endomorphism  $R(X, Y)Z$  of the Lie algebra of vector fields on  $M$ , named the curvature operator is defined by

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z,$$

where  $X, Y$ , and  $Z$  are vector fields on  $M$  and  $[X, Y]$  denotes the Lie bracket of  $X$  and  $Y$ . We also denote  $R(X, Y)Z$  as the derivation induced by the curvature operator. For a symmetric  $(0, 2)$ -tensor  $A$  and any vector fields  $X, Y$ , and  $Z$  on  $M$ , we define the endomorphism  $(X \wedge_A Y)$  of  $M$  by

$$(X \wedge_A Y)Z = A(Y, Z)X - A(X, Z)Y.$$

The Riemannian Christoffel curvature tensor  $R$  and the  $(0, 4)$ -tensor  $G$  are defined by

$$\begin{aligned} R(X, Y, W, Z) &= g(R(X, Y)W, Z), \\ G(X, Y, W, Z) &= g((X \wedge_g Y)Z, W). \end{aligned}$$

respectively.

The  $(0, 6)$ -tensor  $R \cdot R$ , obtained by the action of the curvature operators  $R(X, Y)$  on the  $(0, 4)$ -curvature tensor  $R$ , is given by [9]

(1)

$$\begin{aligned} (R \cdot R)(U, V, W, Z; X, Y) &= -R(R(X, Y)U, V, W, Z) \\ &\quad - R(U, R(X, Y)V, W, Z) - R(U, V, R(X, Y)W, Z) - R(U, V, W, R(X, Y)Z), \end{aligned}$$

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where  $U, V, W, Z, X, Y \in M$ .

The tensor  $R \cdot R$  has the following algebraic properties:

$$\begin{aligned}
 (R \cdot R)(U, V, W, Z; X, Y) &= -(R \cdot R)(V, U, W, Z; X, Y) \\
 &= -(R \cdot R)(U, V, Z, W; X, Y), \\
 (R \cdot R)(U, V, W, Z; X, Y) &+ (R \cdot R)(U, W, Z, V; X, Y) \\
 &+ (R \cdot R)(U, Z, V, W; X, Y) = 0, \\
 (R \cdot R)(U, V, W, Z; X, Y) &= -(R \cdot R)(U, V, W, Z; Y, X), \\
 (R \cdot R)(U, V, W, Z; X, Y) &+ (R \cdot R)(W, Z, X, Y; U, V) \\
 &+ (R \cdot R)(X, Y, U, V; W, Z) = 0.
 \end{aligned}$$

The simplest  $(0, 6)$ -tensor having the same symmetry properties as  $R \cdot R$  may well be the Tachibana tensor  $Q(g, R)$  defined by [9]

$$\begin{aligned}
 (2) \quad Q(g, R)(U, V, W, Z; X, Y) &= R((X \wedge Y)U, V, W, Z) + R(U, (X \wedge Y)V, W, Z) \\
 &+ R(U, V, (X \wedge Y)W, Z) + R(U, V, W, (X \wedge Y)Z)
 \end{aligned}$$

In the context of generalized Sasakian-space-forms, U. K. Kim [7] studied locally symmetric properties of generalized Sasakian-space-forms. In [5] U. C. De and A. Sarkar have studied some symmetry properties of generalized Sasakian-space-forms regarding the projective curvature tensor. In [8] D. G. Prakasha studied some pseudosymmetric properties of generalized Sasakian-space-forms with Weyl conformal curvature tensor. In [3] C. S. Bagewadi and G. Ingalahalli studied some results of C-Bochner curvature tensor and  $\tau$ -curvature tensor of a generalized Sasakian space forms. As a continuation of this, we plan to study generalized Sasakian space forms satisfying certain curvature conditions on Quasi conformal curvature tensor.

## 2. PRELIMINARIES

A  $(2n + 1)$  dimensional Riemannian manifold is called an almost contact metric manifold if the following results hold [4]

$$\begin{aligned}
 \phi^2 X &= -X + \eta(X)\xi, & \phi\xi &= 0, \\
 \eta(\xi) &= 1, & g(X, \xi) &= \eta(X), \\
 g(\phi X, \phi Y) &= g(X, Y) - \eta(X)\eta(Y), & \eta(\phi X) &= 0, \\
 g(\phi X, Y) &= -g(X, \phi Y), & g(\phi X, X) &= 0, \\
 & & (\nabla_X \eta)(Y) &= g(\nabla_X \xi, Y).
 \end{aligned}$$

For a  $(2n + 1)$  dimensional generalized Sasakian-space-forms we have [1]

$$(3) \quad \begin{aligned} R(X, Y)Z &= f_1\{g(Y, Z)X - g(X, Z)Y\} \\ &\quad + f_2\{g(X, \phi Z)\phi Y - g(Y, \phi Z)\phi X + 2g(X, \phi Y)\phi Z\} \\ &\quad + f_3\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X, Z)\eta(Y)\xi - g(Y, Z)\eta(X)\xi\}. \end{aligned}$$

For any vector fields  $X, Y, Z$  on  $M$ , where  $R$  denotes the curvature tensor of  $M$ ,

$$\begin{aligned} S(X, Y) &= (2nf_1 + 3f_2 - f_3)g(X, Y) + (3f_2 - (2n - 1)f_3)\eta(X)\eta(Y), \\ QX &= (2nf_1 + 3f_2 - f_3)X + (3f_2 - (2n - 1)f_3)\eta(X)\xi, \\ r &= 2n(2n + 1)f_1 + 6nf_2 - 4nf_3 \end{aligned}$$

are valid, where  $f_1, f_2, f_3$  are differentiable functions on  $M$  and  $X, Y, Z$  are vector fields on  $M$ . In such case we will write the manifold as  $M(f_1, f_2, f_3)$ . This kind of manifolds appears as natural generalization of the Sasakian-space-forms by taking  $f_1 = \frac{c+3}{4}$  and  $f_2 = f_3 = \frac{c-1}{4}$ , where  $c$  denotes constant  $\phi$ -sectional curvature. The  $\phi$ -sectional curvature of generalized Sasakian-space-forms  $M(f_1, f_2, f_3)$  is  $f_1 + 3f_2$ . Moreover, cosymplectic space-forms and Kenmotsu space-forms are also particular case of generalized Sasakian-space-forms.

For generalized Sasakian-space-forms we also have

$$\begin{aligned} (4) \quad R(X, Y)\xi &= (f_1 - f_3)[\eta(Y)X - \eta(X)Y], \\ R(\xi, X)Y &= (f_1 - f_3)[g(X, Y)\xi - \eta(Y)X], \\ (5) \quad R(\xi, X)\xi &= (f_1 - f_3)[\eta(X)\xi - X], \\ R(\xi, \xi)X &= 0, \\ S(X, \xi) &= 2n(f_1 - f_3)\eta(X). \end{aligned}$$

### 3. SEMI SYMMETRIC GENERALIZED SASAKIAN-SPACE-FORMS

Let  $M(f_1, f_2, f_3)$  be a  $(2n + 1)$  dimensional semi symmetric generalized Sasakian-space-forms. Then from (1) we have

$$\begin{aligned} (6) \quad (R \cdot R)(U, V, W, Z; X, Y) &= 0, \\ (R(X, Y) \cdot R)(U, V, W, Z) &= 0, \\ -R(R(X, Y)U, V, W, Z) - R(U, R(X, Y)V, W, Z) \\ -R(U, V, R(X, Y)W, Z) - R(U, V, W, R(X, Y)Z) &= 0, \\ R(R(X, Y)U, V, W, Z) + R(U, R(X, Y)V, W, Z) \\ +R(U, V, R(X, Y)W, Z) + R(U, V, W, R(X, Y)Z) &= 0. \end{aligned}$$

In view of (4) and (5), for  $X = U = \xi$ , (6) yields

$$(f_1 - f_3)[R(Y, V, W, Z) + (f_1 - f_3)g(Y, W)g(V, Z) - (f_1 - f_3)g(Y, Z)g(V, W)] = 0.$$

Since  $(f_1 - f_3) \neq 0$ , we have

$$(7) \quad R(Y, V, W, Z) = (f_1 - f_3)g(Y, Z)g(V, W) + (f_1 - f_3)g(Y, W)g(V, Z).$$

Let  $\{e_1, e_2, \dots, e_{2n+1}\}$  be an orthonormal basis of the tangent space at each point of the manifold. Putting  $Y = Z = e_i$  in (7) and taking summation over  $i$ ,  $(1 \leq i \leq (2n + 1))$ , we get

$$(8) \quad S(V, W) = 2n(f_1 - f_3)g(V, W).$$

Therefore,  $M(f_1, f_2, f_3)$  is an Einstein manifold. Hence we state the following.

**Theorem 1.** *Let  $M(f_1, f_2, f_3)$  be a  $(2n + 1)$  dimensional generalized Sasakian-space-forms. If  $M(f_1, f_2, f_3)$  is semi symmetric then  $M(f_1, f_2, f_3)$  is an Einstein manifold.*

**Corollary 1.**  *$(g, V, \lambda)$  is Ricci soliton in semi symmetric generalized Sasakian space forms if and only if  $V$  is conformal killing vector field.*

*Proof.* From Theorem 1 and by the definition of Ricci soliton [6], we have

$$(L_V g)(V, W) + 2S(V, W) + 2\lambda g(V, W) = 0,$$

where  $\lambda$  is some constant. From (8) we get

$$(9) \quad (L_V g)(V, W) + 4n(f_1 - f_3)g(V, W) + 2\lambda g(V, W) = 0.$$

Let  $\{e_1, e_2, \dots, e_{2n+1}\}$  be an orthonormal basis of the tangent space at each point of the manifold. Putting  $V = W = e_i$  in (9) and taking summation over  $i$ ,  $(1 \leq i \leq (2n + 1))$ , we get

$$(L_V g)(e_i, e_i) + 4n(2n + 1)(f_1 - f_3) + 2(2n + 1)\lambda = 0.$$

Since  $[e_i, e_j] = 0$ , for all  $1 \leq i, j \leq (2n + 1)$ , we obtain

$$\lambda = -2n(f_1 - f_3).$$

Thus, Ricci soliton in semi symmetric generalized Sasakian-space forms is shrinking if  $f_1 > f_3$ , i.e.,  $\lambda < 0$ , steady if  $f_1 = f_3$ , i.e.,  $\lambda = 0$ , and expands if  $f_1 < f_3$ , i.e.,  $\lambda > 0$ .  $\square$

#### 4. PSEUDOSYMMETRIC GENERALIZED SASAKIAN-SPACE-FORMS

Let  $M(f_1, f_2, f_3)$  be a  $(2n + 1)$  dimensional generalized Sasakian-space-forms. Then from (1) and (2) we have

$$\begin{aligned} (R \cdot R)(U, V, W, Z; X, Y) &= L_R Q(g, R)(U, V, W, Z; X, Y), \\ (R(X, Y) \cdot R)(U, V, W, Z) &= -L_R((X \wedge Y) \cdot R)(U, V, W, Z), \\ (10) \quad & -R(R(X, Y)U, V, W, Z) - R(U, R(X, Y)V, W, Z) - R(U, V, R(X, Y)W, Z) \\ & -R(U, V, W, R(X, Y)Z) = L_R[R((X \wedge Y)U, V, W, Z) + R(U, (X \wedge Y)V, W, Z) \\ & + R(U, V, (X \wedge Y)W, Z) + R(U, V, W, (X \wedge Y)Z)] \end{aligned}$$

In view of (4) and (5), for  $X = U = \xi$ , (10) yields

$$\begin{aligned} & (f_1 - f_3)[R(Y, V, W, Z) + (f_1 - f_3)g(Y, W)g(V, Z) - (f_1 - f_3)g(Y, Z)g(V, W)] \\ &= -L_R[R(Y, V, W, Z) + (f_1 - f_3)g(Y, W)g(V, Z) - (f_1 - f_3)g(Y, Z)g(V, W)], \\ & [L_R + (f_1 - f_3)][R(Y, V, W, Z) + (f_1 - f_3)g(Y, W)g(V, Z) \\ & \quad - (f_1 - f_3)g(Y, Z)g(V, W)] = 0. \end{aligned}$$

Therefore either  $L_R = -(f_1 - f_3)$  or

$$(11) \quad R(Y, V, W, Z) = (f_1 - f_3)g(Y, Z)g(V, W) - (f_1 - f_3)g(Y, W)g(V, Z).$$

Let  $\{e_1, e_2, \dots, e_{2n+1}\}$  be an orthonormal basis of the tangent space at each point of the manifold. Putting  $Y = Z = e_i$  in (11) and taking summation over  $i$ , ( $1 \leq i \leq (2n+1)$ ), we get

$$S(V, W) = 2n(f_1 - f_3)g(V, W).$$

Therefore,  $M(f_1, f_2, f_3)$  is an Einstein manifold. Hence we state the following.

**Theorem 2.** *Let  $M(f_1, f_2, f_3)$  be a  $(2n+1)$  dimensional generalized Sasakian-space-forms. If  $M(f_1, f_2, f_3)$  is pseudosymmetric then  $M(f_1, f_2, f_3)$  is an Einstein manifold provided that  $L_R \neq -(f_1 - f_3)$ .*

**Corollary 2.**  *$(g, V, \lambda)$  is Ricci soliton in pseudo symmetric generalized Sasakian space forms if and only if  $V$  is conformal killing vector field provided  $L_R \neq -(f_1 - f_3)$ .*

*Proof.* Proof follows from Theorem 2 and the definition of Ricci Soliton.  $\square$

## 5. QUASI CONFORMAL SEMI SYMMETRIC GENERALIZED SASAKIAN-SPACE-FORMS

For a  $(2n+1)$  dimensional almost contact metric manifold the Quasi conformal curvature tensor  $C$  is given by

$$\begin{aligned} & C(X, Y)Z \\ (12) \quad &= aR(X, Y)Z + b[S(Y, Z)X - S(X, Z)Y + g(Y, Z)QX - g(X, Z)QY] \\ & \quad - \frac{r}{2n+1} \left[ \frac{a}{2n} + 2b \right] [g(Y, Z)X - g(X, Z)Y]. \end{aligned}$$

Using equation (3) in (12) yields

$$\begin{aligned} & C(X, Y)\xi = D[\eta(Y)X - \eta(X)Y], \\ (13) \quad & C(\xi, X)Y = D[g(X, Y)\xi - \eta(X)Y], \\ (14) \quad & C(\xi, X)\xi = D[\eta(X)\xi - X], \end{aligned}$$

where

$$D = a(f_1 - f_3) + 2nb(f_1 - f_3) + b(2nf_1 + 3f_2 - f_3) - \frac{r}{2n+1} \left[ \frac{a}{2n} + 2b \right].$$

Let  $M(f_1, f_2, f_3)$  be a  $(2n + 1)$  dimensional Quasi conformal semi symmetric generalized Sasakian-space-forms. Then, from (1), we have

$$\begin{aligned} (R(X, Y) \cdot C)(U, V, W, Z) &= 0, \\ -C(R(X, Y)U, V, W, Z) - C(U, R(X, Y)V, W, Z) \\ -C(U, V, R(X, Y)W, Z) - C(U, V, W, R(X, Y)Z) &= 0. \end{aligned}$$

In view of (4), (5), and (13), for  $X = U = \xi$ , (14) yields

$$(f_1 - f_3)\{C(Y, V, W, Z) + D[g(Y, W)g(V, Z) - g(Y, Z)g(V, W)]\} = 0.$$

Since  $(f_1 - f_3) \neq 0$ , we have

$$(15) \quad C(Y, V, W, Z) = D[g(Y, Z)g(V, W) - g(Y, W)g(V, Z)].$$

Let  $\{e_1, e_2, \dots, e_{2n+1}\}$  be an orthonormal basis of the tangent space at each point of the manifold. Putting  $Y = Z = e_i$  in (15) and taking summation over  $i$ ,  $(1 \leq i \leq (2n + 1))$ , using equation (12), we get

$$S(V, W) = D'g(V, W),$$

where

$$D' = \frac{2n[(a + 2nb)(f_1 - f_3) + b(2nf_1 + 3f_2 - f_3)] - br}{a + b(2n + 1)}.$$

**Theorem 3.** *Let  $M(f_1, f_2, f_3)$  be a  $(2n + 1)$  dimensional generalized Sasakian-space-forms. If  $M(f_1, f_2, f_3)$  is Quasi conformal semisymmetric then  $M(f_1, f_2, f_3)$  is an Einstein manifold.*

## 6. QUASI CONFORMAL PSEUDO SYMMETRIC GENERALIZED SASAKIAN-SPACE-FORMS

Let  $M(f_1, f_2, f_3)$  be a  $(2n + 1)$  dimensional Quasi conformal pseudo symmetric generalized Sasakian-space-forms. Then, from (1) and (2), we have

$$\begin{aligned} (R \cdot C)(U, V, W, Z; X, Y) &= L_C Q(g, C)(U, V, W, Z; X, Y), \\ (R(X, Y) \cdot C)(U, V, W, Z) &= -L_C((X \wedge Y) \cdot R)(U, V, W, Z), \\ (16) \quad -C(R(X, Y)U, V, W, Z) - C(U, R(X, Y)V, W, Z) - C(U, V, R(X, Y)W, Z) \\ -C(U, V, W, R(X, Y)Z) &= L_C[C((X \wedge Y)U, V, W, Z) + C(U, (X \wedge Y)V, W, Z) \\ +C(U, V, (X \wedge Y)W, Z) + C(U, V, W, (X \wedge Y)Z) \end{aligned}$$

In view of (4), (5), and (13), for  $X = U = \xi$ , (16) yields

$$[L_C + (f_1 - f_3)]\{C(Y, V, W, Z) + D[g(Y, W)g(V, Z) - g(Y, Z)g(V, W)]\} = 0.$$

Therefore, either  $L_C = -(f_1 - f_3)$  or

$$(17) \quad C(Y, V, W, Z) = D[g(Y, Z)g(V, W) - g(Y, W)g(V, Z)].$$

Let  $\{e_1, e_2, \dots, e_{2n+1}\}$  be an orthonormal basis of the tangent space at each point of the manifold. Putting  $Y = Z = e_i$  in (17) and taking summation over  $i$ , ( $1 \leq i \leq (2n+1)$ ), using equation (12), we get

$$(18) \quad S(V, W) = D'g(V, W).$$

**Theorem 4.** *Let  $M(f_1, f_2, f_3)$  be a  $(2n+1)$  dimensional generalized Sasakian-space-forms. If  $M(f_1, f_2, f_3)$  is Quasi conformal pseudo symmetric then  $M(f_1, f_2, f_3)$  is an Einstein manifold provided that  $L_C \neq -(f_1 - f_3)$ .*

#### 7. GENERALIZED SASAKIAN-SPACE-FORMS SATISFIES THE CONDITION

$$C \cdot C = 0$$

Let  $M(f_1, f_2, f_3)$  be a  $(2n+1)$  dimensional generalized Sasakian-space-forms. Let  $C \cdot C$  be a  $(0, 6)$ -tensor and  $C \cdot C = 0$ . Then

$$(19) \quad \begin{aligned} &(C(X, Y) \cdot C)(U, V, W, Z) = 0, \\ &-C(C(X, Y)U, V, W, Z) - C(U, C(X, Y)V, W, Z) \\ &-C(U, V, C(X, Y)W, Z) - C(U, V, W, C(X, Y)Z) = 0. \end{aligned}$$

In view of (13) and (14), for  $X = U = \xi$ , (19) yields

$$-D[C(Y, V, W, Z) + D\{g(Y, W)g(V, Z) - g(Y, Z)g(V, W)\}] = 0.$$

Since  $D \neq 0$ , we have

$$(20) \quad C(Y, V, W, Z) = D[g(Y, Z)g(V, W) - g(Y, W)g(V, Z)].$$

Let  $\{e_1, e_2, \dots, e_{2n+1}\}$  be an orthonormal basis of the tangent space at each point of the manifold. Putting  $Y = Z = e_i$  in (20) and taking summation over  $i$ , ( $1 \leq i \leq (2n+1)$ ), using equation (12), we get

$$S(V, W) = D'g(V, W).$$

**Theorem 5.** *Let  $M(f_1, f_2, f_3)$  be a  $(2n+1)$  dimensional generalized Sasakian-space-forms. If  $(0, 6)$ -tensor  $C \cdot C = 0$  holds on  $M(f_1, f_2, f_3)$ , then  $M(f_1, f_2, f_3)$  is an Einstein manifold.*

**Corollary 3.** *Let  $M(f_1, f_2, f_3)$  be a  $(2n+1)$  dimensional Quasi conformal semi symmetric generalized Sasakian-space-forms. Then  $R \cdot C = C \cdot C$  holds on  $M(f_1, f_2, f_3)$ .*

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