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CERTAIN IDENTITIES INVOLVING k-BALANCING AND k-LUCAS-BALANCING NUMBERS VIA MATRICES

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Abstract. Matrix methods are useful to derive several identities for balancing numbers and their related sequences. In this article, two matrices with arithmetic indexes, namely

$$X_a = \begin{pmatrix} 2C_{k,a} & -1 \\ 1 & 0 \end{pmatrix} \qquad \text{and} \qquad Y_a = \begin{pmatrix} C_{k,a} & C_{k,a}^2 - 1 \\ 1 & C_{k,a} \end{pmatrix}$$

are used to derive some identities including certain sum formulas involving k-balancing and k-Lucas-balancing numbers.

1. Introduction

Balancing numbers B and balancers R are solutions of a Diophantine equation $1+2+3+\cdots+(B-1) = (B+1)+(B+2)+\cdots+(B+R)$ and satisfy the linear recurrence $B_{n+1} = 6B_n - B_{n-1}$, $n \geq 2$ with initial conditions $B_0 = 0$ and $B_1 = 1$, where B_n denotes the n^{th} balancing number [1, 3]. They also satisfy the non linear recurrence $B_n^2 - B_{n+1}B_{n-1} = 1$ which we call Cassini formula for balancing numbers. A number sequence very closely associates with balancing numbers is the sequence of Lucas-balancing numbers [9]. For each balancing number B_n, a Lucas-balancing number C_n is defined by $C_n = \sqrt{8B_n^2+1}$. The first few Lucas-balancing numbers are $\{1, 3, 17, 99, 577, \dots\}$ and satisfy the recurrence relation same as that of balancing numbers but with different initials, i.e., $C_{n+1} = 6C_n - C_{n-1}$ with $C_0 = 1$ and $C_1 = 3$. Several interesting identities among balancing and Lucas-balancing numbers were developed in [7]. For instance, the identity resembles with trigonometric identity $\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$ is

$$B_{m\pm n} = B_m C_n \pm C_m B_n$$

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and the identity resembles with D'Movier's theorem is $(C_n + \sqrt{8}B_n)^m = C_{mn} + \sqrt{8}B_{mn}$. Another number sequence known as the sequence of cobalancing numbers was obtained by slightly modifying the original Diophantine equation. Cobalancing numbers b and the cobalancers r are solutions of a Diophantine equation $1+2+3+\cdots+b = (b+1)+(b+2)+\cdots+(b+r)$ [9]. An interesting observation found in [9] is that "Every balancer is a cobalancing number and every cobalancer is a balancing number".

Balancing numbers are generalized in many ways. One of the most general extension of balancing numbers was due to Liptai et al. [6]. He has replaced the original definition of balancing numbers by the following

$$1^k + 2^k + \cdots + (x-1)^k = (x+1)^l + \cdots + (y-1)^l\,, \tag{1}$$

where the exponents k and l are given positive integers. In the work of Liptai et al. [6], effective and non-effective finiteness theorems on (1) are proved. A balancing problem of ordinary binomial coefficients was studied by Komatsu and Szalay [4]. Some more results on generalization of balancing numbers can be seen in [2, 5, 8, 11].

Recently, Ray has studied a one-parameter generalization of balancing numbers known as k-balancing numbers [12]. He defined the k-balancing sequence $\{B_{k,n}\}_n \in N$, $(k \geq 1)$ recursively by $B_{k,n+1} = 6B_{k,n} - B_{k,n-1}$ with $B_{k,0} = 0$, $B_{k,1} = 1$ and $n \geq 1$. First few k-balancing numbers are $\{0, 1, 6k, 36k^2 - 1, 216k^3 - 12k, \ldots, \}$. It is observed that for $k = 1$ the usual balancing numbers are obtained. Similarly, the sequence of k-Lucas-balancing numbers $\{C_{k,n}\}_n \in N$ defined recursively by $C_{k,n+1} = 6C_{k,n} - C_{k,n-1}$ with $C_{k,0} = 1$, $C_{k,1} = 3k$ and usual Lucas-balancing numbers are obtained for $k = 1$. The Binet's formulas for both k-balancing and k-Lucas-balancing numbers are respectively given by $B_{k,n} = \frac{\lambda_1^n - \lambda_2^n}{\lambda 1 - \lambda 2}$ and $C_{k,n} = \frac{\lambda_1^n + \lambda_2^n}{2}$, where $\lambda_1 = 3k + \sqrt{9k^2 - 1}$ and $\lambda_2 = 3k - \sqrt{9k^2 - 1}$. Notice that $\lambda_1 + \lambda_2 = 6k$ and $\lambda_1\lambda_2 = 1$. Several identities concerning k-balancing and k-Lucas-balancing numbers can be found in [12, 13]. Few of them are summarized below which will be needed later.

$$\begin{aligned} B_{k,n+1} - B_{k,n-1} &= 2C_{k,n}, & B_{k,-n} &= -B_{k,n}, \\ B_{k,n}^2 - B_{k,n+1}B_{k,n-1} &= 1, & B_{k,2n} &= 2B_{k,n}C_{k,n}. \end{aligned}$$

Matrix methods are useful tools to derive identities for balancing numbers and their related number sequences [11]. In this article, some k-balancing and k-Lucas-balancing sums with arithmetic indexes, say $an+r$ with fixed integers a and r with $0 \leq r \leq a-1$, are derived using the matrices

$$X_a = \begin{pmatrix} 2C_{k,a} & -1 \\ 1 & 0 \end{pmatrix} \qquad \text{and} \qquad Y_a = \begin{pmatrix} C_{k,a} & C_{k,a}^2 - 1 \\ 1 & C_{k,a} \end{pmatrix}.$$

2. Some identities of k-balancing numbers via matrices

In this section, some known and new identities concerning k-balancing numbers are obtained using matrices. Before doing this, we need the following result.

Theorem 1. *Let M be a square matrix with $M^2 = 2C_{k,a}M - I$ where a is a fixed positive integer and I denotes the identity matrix of order 2. Then*

$$M^n = \frac{1}{B_{k,a}}\left[B_{k,an}M - B_{k,a(n-1)}I\right]$$

for all integers n.

Proof. Since n is any integer, the following three cases will arise. The first case for $n = 0$ is obvious.

In order to prove the second case, i.e., for $n \geq 1$, we use principle of induction. The basis step is clear for $n = 1$. Assume that, the result is true for all $m \leq n$. Then by inductive hypothesis,

$$M^m = \frac{1}{B_{k,a}}\left[B_{k,am}M - B_{k,a(m-1)}I\right].$$

Now, in the inductive step,

$$\begin{aligned} M^{m+1} &= M^m M \\ &= \frac{1}{B_{k,a}}\left[B_{k,am}M - B_{k,a(m-1)}I\right]M \\ &= \frac{1}{B_{k,a}}\left[B_{k,am}M^2 - B_{k,a(m-1)}M\right] \\ &= \frac{1}{B_{k,a}}\left[B_{k,am}(2C_{k,a}M - I) - B_{k,a(m-1)}M\right] \\ &= \frac{1}{B_{k,a}}\left[(2C_{k,a}B_{k,am} - B_{k,a(m-1)})M - B_{k,am}IM\right]. \end{aligned}$$

The required result follows as $2C_{k,a}B_{k,am} - B_{k,a(m-1)} = B_{k,a(m+1)}$.

In case 3, we need to show $M^{-n} = \frac{1}{B_{k,a}}\left[B_{k,-an}M - B_{k,a(-n-1)}I\right]$. For that, let $A = 2C_{k,a}I - M = M^{-1}$. Then

$$\begin{aligned} A^2 &= 4C_{k,a}^2 I - 4C_{k,a}M + M^2 \\ &= 2C_{k,a}(2C_{k,a}I - M) + (M^2 - 2C_{k,a}M) \\ &= 2C_{k,a}A - I. \end{aligned}$$

Therefore, by case 2, $A^n = \frac{1}{B_{k,a}}\left[B_{k,an}A - B_{k,a(n-1)}I\right]$. It follows that

$$\begin{aligned} B_{k,a}M^{-n} &= B_{k,an}(2C_{k,a}I - M) - B_{k,a(n-1)}I \\ &= -B_{k,an}M + (2C_{k,a}B_{k,an} - B_{a(n-1)})I \\ &= -B_{k,an}M + B_{k,a(n+1)}, \end{aligned}$$

and the proof completes as $B_{k,-n} = -B_{k,n}$. $\square$

Now let us introduce a second order matrix $X_a = \begin{pmatrix} 2C_{k,a} & -1 \\ 1 & 0 \end{pmatrix}$ and using induction and the result of Theorem 1, we observe that, for any integer $n \geq 1$,

$$X_a^n = \frac{1}{B_{k,a}} \begin{pmatrix} B_{k,a(n+1)} & -B_{k,an} \\ B_{k,an} & -B_{k,a(n-1)} \end{pmatrix}.$$

It is also noticed that the matrix X_a^n satisfies the recurrence relation $X_a^{n+1} = 2C_{k,a}X_a^n - X_a^{n-1}$ for $n \geq 1$ and with initials $X_a^0 = I$ and $X_a^1 = X_a$. Furthermore, let us define another second order matrix $Y_a = \begin{pmatrix} C_{k,a} & C_{k,a}^2 - 1 \\ 1 & C_{k,a} \end{pmatrix}$ and we prove the following result.

Lemma 1. *Let* $Y_a = \begin{pmatrix} C_{k,a} & C_{k,a}^2 - 1 \\ 1 & C_{k,a} \end{pmatrix}$. *Then, for* $n \geq 1$,

$$Y_a^n = \frac{1}{B_{k,a}} \begin{pmatrix} B_{k,a(n+1)} - C_{k,a}B_{k,an} & (C_{k,a}^2 - 1)B_{k,an} \\ B_{k,an} & B_{k,a(n+1)} - C_{k,a}B_{k,an} \end{pmatrix}.$$

Proof. Method of induction is used to prove this result. The result is obvious for $n = 1$. Assume that

$$Y_a^{n-1} = \frac{1}{B_{k,a}} \begin{pmatrix} B_{k,an} - C_{k,a}B_{k,a(n-1)} & (C_{k,a}^2 - 1)B_{k,a(n-1)} \\ B_{k,a(n-1)} & B_{k,an} - C_{k,a}B_{k,a(n-1)} \end{pmatrix}.$$

Now in the inductive step,

$$\begin{aligned} Y_a^n &= Y_a^{n-1}Y_a \\ &= \frac{1}{B_{k,a}} \begin{pmatrix} B_{k,an} - C_{k,a}B_{k,a(n-1)} & (C_{k,a}^2 - 1)B_{k,a(n-1)} \\ B_{k,a(n-1)} & B_{k,an} - C_{k,a}B_{k,a(n-1)} \end{pmatrix} \begin{pmatrix} C_{k,a} & C_{k,a}^2 - 1 \\ 1 & C_{k,a} \end{pmatrix}. \end{aligned}$$

By usual matrix multiplication and after some algebraic manipulation, we obtain the desired result. $\square$

It is seen that $\det Y_a = 1$ implies that $\det Y_a^n = 1$. That is,

$$\frac{1}{B_{k,a}^2} \left[(B_{k,a(n+1)} - C_{k,a}B_{k,an})^2 - (C_{k,a}^2 - 1)B_{k,an}^2 \right] = 1,$$

and the following result will obtain.

Lemma 2. *For any integer* $n \geq 1$,

$$B_{k,a(n+1)}^2 - 2C_{k,a}B_{k,a(n+1)}B_{k,an} + B_{k,an}^2 = B_{k,a}^2.$$

The following are two fundamental identities concerning k-balancing and k-Lucas-balancing numbers that are obtained by using the matrix Y_a.

Theorem 2. *For all natural numbers* n *and* m,

$$B_{k,a}B_{k,a(n+m)} = B_{k,a(n+1)}B_{k,am} - 2C_{k,a}B_{k,an}B_{k,am} + B_{k,a(m+1)}B_{k,an}.$$

Proof. For any natural numbers n and m,

$$Y_a^{n+m} = \frac{1}{B_{k,a}} \begin{pmatrix} B_{k,a(n+m+1)} - C_{k,a}B_{k,a(n+m)} & (C_{k,a}^2 - 1)B_{k,a(n+m)} \\ B_{k,a(n+m)} & B_{k,a(n+m+1)} - C_{k,a}B_{k,a(n+m)} \end{pmatrix}.$$

On the other hand,

$$Y_a^n Y_a^m = \frac{1}{B_{k,a}^2} \begin{pmatrix} B_{k,a(n+1)} - C_{k,a}B_{k,an} & (C_{k,a}^2 - 1)B_{k,an} \\ B_{k,an} & B_{k,a(n+1)} - C_{k,a}B_{k,an} \end{pmatrix}$$
$$\begin{pmatrix} B_{k,a(m+1)} - C_{k,a}B_{k,am} & (C_{k,a}^2 - 1)B_{k,am} \\ B_{k,am} & B_{k,a(m+1)} - C_{k,a}B_{k,am} \end{pmatrix}.$$

Since $Y_a^{n+m} = Y_a^n Y_a^m$ and comparing the $(2,1)$ entries from both sides of the matrices, the desired result is obtained. □

The following is an immediate consequence of Theorem 2.

Corollary 1. *For any natural numbers m and n, $B_{k,n+m} = B_{k,n+1}B_{k,m} - B_{k,n}B_{k,m-1}$.*

Proof. Putting $a = 1$ in the result of Theorem 2 and using the identity $B_{k,m+1} - 2Ck, aB_{k,am} = -B_{k,m-1}$, we obtain the desired result. □

Theorem 3. *For all natural numbers n and m,*

$$B_{k,a}B_{k,a(n-m)} = B_{k,a(m+1)}B_{k,an} - B_{k,a(n+1)}B_{k,am}.$$

Proof. Since $Y_a^{n-m} = Y_a^n[Y_a^m]^{-1}$, proceed similarly as in Theorem 2, we get the required identity. □

In particular for $a = 1$, we have the following corollary.

Corollary 2. *For any natural numbers m and n, $B_{k,n-m} = B_{k,m+1}B_{k,n} - B_{k,n+1}B_{k,m}$.*

3. Sum formulas for k-balancing numbers with rational index

In this section, we derive certain sum formulas for k-balancing numbers with rational index, in particular of the kind an, where a is a positive integer. We use the matrix Y_a to establish these results.

Theorem 4. *Let n be any integer and a be any positive integer. Then*

$$\sum_{j=0}^{n} B_{k,aj} = \frac{B_{k,a(n+1)} - B_{k,an} - B_{k,a}}{B_{k,a+1} - B_{k,a-1} - 2}.$$

Proof. For any integer n and $a \geq 1$, $I - Y_a^{n+1} = (I - Y_a)\sum_{j=0}^{n} Y_a^j$, where I is the 2×2 identity matrix. It follows that

$$\sum_{j=0}^{n} Y_a^j = (I - Y_a)^{-1}(I - Y_a^{n+1}). \tag{2}$$

In fact, $(I-Y_a)^{-1}$ exists since $\det(I-Y_a)=2-2C_{k,a}\neq 0$. Equation (2) can be rewritten as

$$\frac{1}{B_{k,a}}\begin{pmatrix}\sum_{j=0}^{n} B_{k,a(j+1)}-C_{k,a}B_{k,aj} & \sum_{j=0}^{n}(C_{k,a}^2-1)B_{k,aj}\\ \sum_{j=0}^{n} B_{k,aj} & \sum_{j=0}^{n} B_{k,a(j+1)}-C_{k,a}B_{k,aj}\end{pmatrix}$$
$$=\frac{1}{(2-2C_{k,a})B_{k,a}}\begin{pmatrix}1-C_{k,a} & C_{k,a}^2-1\\ 1 & 1-C_{k,a}\end{pmatrix}$$
$$\begin{pmatrix}B_{k,a}-B_{k,a(n+2)}-C_{k,a}B_{k,a(n+1)} & (C_{k,a}^2-1)B_{k,a(n+1)}\\ B_{k,a(n+1)} & B_{k,a}-B_{k,a(n+2)}-C_{k,a}B_{k,a(n+1)}\end{pmatrix}.$$

Performing usual matrix multiplication on right hand side of the above identity, using the formulas $2C_{k,a}-2=B_{k,a+1}-B_{k,a-1}-2$, $B_{k,m+n}+B_{k,m-n}=2B_{k,m}Ck,n$ and some algebraic manipulation, we get the desired result. □

Theorem 5. *Let n be any integer and a be any positive integer. Then*

$$\sum_{j=0}^{n}(-1)^j B_{k,aj}=\frac{B_{k,a(n+1)}+B_{k,an}-B_{k,a}}{B_{k,a+1}-B_{k,a-1}+2}.$$

Proof. For any even integer n and $a\geq 1$, $I+Y_a^{n+1}=(I+Y_a)\sum_{j=0}^{n}(-1)^jY_a^j$, hence

$$\sum_{j=0}^{n}(-1)^jY_a^j=(I+Y_a)^{-1}(I+Y_a^{n+1}). \tag{3}$$

The inverse $(I+Y_a)^{-1}$ surely exists because $\det(I+Y_a)=2+2C_{k,a}\neq 0$. We rewrite (3) as

$$\frac{1}{B_{k,a}}\begin{pmatrix}\sum_{j=0}^{n}(-1)^j[B_{k,a(j+1)}-C_{k,a}B_{k,aj}] & \sum_{j=0}^{n}(-1)^j(C_{k,a}^2-1)B_{k,aj}\\ \sum_{j=0}^{n}(-1)^jB_{k,aj} & \sum_{j=0}^{n}(-1)^j[B_{k,a(j+1)}-C_{k,a}B_{k,aj}]\end{pmatrix}$$
$$=\frac{1}{(2+2C_{k,a})B_{k,a}}\begin{pmatrix}1+C_{k,a} & -(C_{k,a}^2-1)\\ -1 & 1+C_{k,a}\end{pmatrix}$$
$$\begin{pmatrix}B_{k,a}+B_{k,a(n+2)}-C_{k,a}B_{k,a(n+1)} & (C_{k,a}^2-1)B_{k,a(n+1)}\\ B_{k,a(n+1)} & B_{k,a}+B_{k,a(n+2)}-C_{k,a}B_{k,a(n+1)}\end{pmatrix}.$$

Performing usual matrix multiplication on right hand side of the above identity, using the formulas $2C_{k,a}-2=B_{k,a+1}-B_{k,a-1}-2$, $B_{k,m+n}+B_{k,m-n}=2B_{k,m}Ck,n$ and some algebraic manipulation, we get the desired result. □

4. Identities involving k-balancing and k-Lucas-balancing numbers using matrices

In this section, some special relations between matrices and k-balancing and k-Lucas-balancing numbers are investigated. This investigation allows us to establish new and some known identities concerning k-balancing and k-Lucas-balancing numbers.

Theorem 6. *If X is a square matrix with $X^2 = 6kX - I$, where I is the identity matrix with the same order as X, then for all integers n, $X^n = B_{k,n}X - B_{k,n-1}I$.*

Proof. There are three possibilities for n, either $n = 0$ or $n \in Z^+$ or $n \in Z^-$. The result is clearly true for the first case, i.e., for $n = 0$.

For positive integers n, we use the mathematical induction method to prove the result. Clearly, the result is true for $n = 1$ as $X^1 = B_{k,1}X - B_{k,0}I = X$. Assume that the result is true for all n. Then, by inductive hypothesis, $X^n = B_{k,n}X - B_{k,n-1}I$. Proceeding to inductive step, using the recurrence relation for k-balancing numbers and the fact $X^2 = 6kX - I$, we have

$$\begin{aligned} B_{k,n+1}X - B_{k,n}I &= (6kXB_{k,n} - B_{k,n-1}X) - B_{k,n}I \\ &= (6kX - I)B_{k,n} - B_{k,n-1}X \\ &= X^2B_{k,n} - B_{k,n-1}X \\ &= (B_{k,n}X - B_{k,n-1}I)X. \end{aligned}$$

Using the inductive hypothesis, we have $B_{k,n+1}X - B_{k,n}I = X^{n+1}$ and the result follows. Now to finish the proof, we need to show that, for all natural number n, $X^{-n} = B_{k,-n}X - B_{k,-n-1}I$. For that, let $Y = 6kI - X = X^{-1}$, then $Y^2 = 36k^2I - 12kIX + X^2$. Since $X^2 = 6kX - I$, it follows that $Y^2 = 6k(6kI - X) - I$. Further simplification gives $Y^2 = 6kY - I$ and hence $Y^n = B_{k,n}Y - B_{k,n-1}I$. As $Y = 6kI - X = X^{-1}$, this identity reduces to $X^{-n} = (6k_{k,n} - B_{k,n-1})I - B_{k,n}X = B_{k,n+1}I - B_{k,n}X$. The proof completes as $B_{k,n} = -B_{k,-n}$. □

Corollary 3. *If the k-balancing matrix is $M = \begin{pmatrix} 6k & -1 \\ 1 & 0 \end{pmatrix}$, then*

$$M^n = \begin{pmatrix} B_{k,n+1} & -B_{k,n} \\ B_{k,n} & B_{k,n-1} \end{pmatrix},$$

for every integer n.

Proof. Since $M^2 = 6kM - I$,

$$M^n = B_{k,n}M - B_{k,n-1}I = \begin{pmatrix} 6kB_{k,n} & -B_{k,n} \\ B_{k,n} & 0 \end{pmatrix} - \begin{pmatrix} B_{k,n-1} & 0 \\ 0 & B_{k,n-1} \end{pmatrix},$$

and the result follows. □

Corollary 4. *Let* $T = \begin{pmatrix} 3k & 9k^2-1 \\ 1 & 3k \end{pmatrix}$*, then* $T^n = \begin{pmatrix} C_{k,n} & (9k^2-1)B_{k,n} \\ B_{k,n} & C_{k,n} \end{pmatrix}$*, for every integer* n*.*

Proof. The proof is similar to the proof of Corollary 3. □

Lemma 3. *For every integer* n*,* $C_{k,n}^2 - (9k^2-1)B_{k,n}^2 = 1$*.*

Proof. It is observed that $\det T = 1$. It follows that $\det T^n = 1$. Consequently, $\det \begin{pmatrix} C_{k,n} & (9k^2-1)B_{k,n} \\ B_{k,n} & C_{k,n} \end{pmatrix} = 1$, and the result follows. □

Lemma 4. *For all integers* m *and* n*,* $C_{k,m+n} = C_{k,m}C_{k,n} + (9k^2-1)B_{k,m}B_{k,n}$ *and* $B_{k,m+n} = B_{k,m}C_{k,n} + C_{k,m}B_{k,n}$*.*

Proof. For all integers m and n,

$$\begin{aligned} T^{m+n} &= T^m T^n \\ &= \begin{pmatrix} C_{k,m} & (9k^2-1)B_{k,m} \\ B_{k,m} & C_{k,m} \end{pmatrix} \begin{pmatrix} C_{k,n} & (9k^2-1)B_{k,n} \\ B_{k,n} & C_{k,n} \end{pmatrix} \\ &= \begin{pmatrix} C_{k,m}C_{k,n} + (9k^2-1)B_{k,m}B_{k,n} & (9k^2-1)(B_{k,m}C_{k,n} + C_{k,m}B_{k,n}) \\ B_{k,m}C_{k,n} + C_{k,m}B_{k,n} & C_{k,m}C_{k,n} + (9k^2-1)B_{k,m}B_{k,n} \end{pmatrix}. \end{aligned}$$

On the other hand,

$$T^{m+n} = \begin{pmatrix} C_{k,m+n} & (9k^2-1)B_{k,m+n} \\ B_{k,m+n} & C_{k,m+n} \end{pmatrix}.$$

The desired results are obtained by equating the corresponding entries from both matrices. □

Lemma 5. *For all integers* m *and* n*,* $C_{k,m-n} = C_{k,m}C_{k,n} - (9k^2-1)B_{k,m}B_{k,n}$ *and* $B_{k,m-n} = B_{k,m}C_{k,n} - C_{k,m}B_{k,n}$*.*

Proof. The proof of this result is analogous to the previous proof. □

The following results directly follow from Lemma 1 and Lemma 2.

Lemma 6. *For all integers* m *and* n*,* $C_{k,m+n} + C_{k,m-n} = 2C_{k,m}C_{k,n}$ *and* $B_{k,m+n} + B_{k,m-n} = 2B_{k,m}C_{k,n}$*.*

Lemma 7. *For all integers* $x, y,$ *and* z*,*

$$B_{k,x+y+z} = B_{k,x}C_{k,y}C_{k,z} + C_{k,x}B_{k,y}C_{k,z} + C_{k,x}C_{k,y}B_{k,z} + (9k^2-1)B_{k,x}C_{k,y}B_{k,z},$$

and

$$C_{k,x+y+z} = C_{k,x}C_{k,y}C_{k,z} + (9k^2-1)\left[B_{k,x}B_{k,y}C_{k,z} + B_{k,x}C_{k,y}B_{k,z} + C_{k,x}B_{k,y}B_{k,z}\right].$$

Proof. For all integers $x, y,$ and z,

$$T^{x+y+z} = \begin{pmatrix} C_{k,x+y+z} & (9k^2-1)B_{k,x+y+z} \\ B_{k,x+y+z} & C_{k,x+y+z} \end{pmatrix}.$$

On the other hand,

$$\begin{aligned}T^{x+y+z} &= T^{x+y}T^{z}\\ &= \begin{pmatrix} C_{k,x+y} & (9k^2-1)B_{k,x+y} \\ B_{k,x+y} & C_{k,x+y} \end{pmatrix} \begin{pmatrix} C_{k,z} & (9k^2-1)B_{k,z} \\ B_{k,z} & C_{k,z} \end{pmatrix}.\end{aligned}$$

Putting the values of $C_{k,x+y}$ and $B_{k,x+y}$ using Lemma 4, performing the matrix multiplication and equating the corresponding entries of the matrices, we obtain the desired results. □

Lemma 8. *For all integers x, y, and z, $C^2_{k,x+y} - 2(9k^2-1)C_{k,x+y}B_{k,y+z}B_{k,z-x} - (9k^2-1)B^2_{k,y+z} = C^2_{k,z-x}$.*

Proof. Consider the following matrix multiplication:

$$\begin{pmatrix} C_{k,x} & (9k^2-1)B_{k,x} \\ B_{k,z} & C_{k,z} \end{pmatrix} \begin{pmatrix} C_{k,y} \\ B_{k,y} \end{pmatrix} = \begin{pmatrix} C_{k,x+y} \\ C_{k,y+z} \end{pmatrix}.$$

Using Lemma 5, $\det \begin{pmatrix} C_{k,x} & (9k^2-1)B_{k,x} \\ B_{k,z} & C_{k,z} \end{pmatrix} = C_{k,z-x} \neq 0$ and therefore, we obtain

$$\begin{pmatrix} C_{k,y} \\ B_{k,y} \end{pmatrix} = \frac{1}{C_{k,z-x}} \begin{pmatrix} C_{k,z} & -(9k^2-1)B_{k,x} \\ -B_{k,z} & C_{k,x} \end{pmatrix} \begin{pmatrix} C_{k,x+y} \\ B_{k,y+z} \end{pmatrix}.$$

It follows that

$$C_{k,y} = \frac{C_{k,z}C_{k,x+y} - (9k^2-1)B_{k,x}B_{k,y+z}}{C_{k,z-x}},$$

and

$$B_{k,y} = \frac{C_{k,x}B_{k,y+z} - C_{k,x+y}B_{k,z}}{C_{k,z-x}}.$$

By virtue of Lemma 3, $C^2_{k,y} - (9k^2-1)B^2_{k,y} = 1$. Putting the values of $C_{k,y}$ and $B_{k,y}$ in this identity and after some algebraic manipulation, we obtain

$$\begin{aligned}C^2_{k,x+y}[C^2_{k,z} - (9k^2-1)B^2_{k,z}] - 2(9k^2-1)C_{k,x+y}B_{k,y+z}[C_{k,x}B_{k,z} - B_{k,x}C_{k,z}]\\ -(9k^2-1)B^2_{k,y+z}[C^2_{k,x} - (9k^2-1)B^2_{k,x}] = C^2_{k,z-x}.\end{aligned}$$

Using Lemma 3 and Lemma 6, we obtain the desired result. □

Lemma 9. *For all integers x, y, and z*

$$C^2_{k,x+y} - C_{k,x+y}C_{k,y+z}C_{k,z-x} - C^2_{k,y+z} = (9k^2-1)B^2_{k,z-x},$$

where $x \neq z$.

Proof. Consider the following matrix multiplication:

$$\begin{pmatrix} C_{k,x} & (9k^2-1)B_{k,x} \\ C_{k,z} & (9k^2-1)B_{k,z} \end{pmatrix} \begin{pmatrix} C_{k,y} \\ B_{k,y} \end{pmatrix} = \begin{pmatrix} C_{k,x+y} \\ C_{k,y+z} \end{pmatrix}.$$

Since $\det \begin{pmatrix} C_{k,x} & (9k^2-1)B_{k,x} \\ C_{k,z} & (9k^2-1)B_{k,z} \end{pmatrix} = (9k^2-1)B_{k,z-x} \neq 0$ for $x \neq z$, we have

$$\begin{pmatrix} C_{k,y} \\ B_{k,y} \end{pmatrix} = \frac{1}{(9k^2-1)B_{k,z-x}} \begin{pmatrix} (9k^2-1)B_{k,z} & -(9k^2-1)B_{k,x} \\ -C_{k,z} & C_{k,x} \end{pmatrix} \begin{pmatrix} C_{k,x+y} \\ B_{k,y+z} \end{pmatrix}.$$

It follows that

$$C_{k,y} = \frac{B_{k,z}C_{k,x+y} - B_{k,x}C_{k,y+z}}{B_{k,z-x}}$$

and

$$B_{k,y} = \frac{C_{k,x}C_{k,y+z} - C_{k,x+y}C_{k,z}}{(9k^2-1)B_{k,z-x}}.$$

Putting the values of $C_{k,y}$ and $B_{k,y}$ in the identity $C_{k,y}^2 - (9k^2-1)B_{k,y}^2 = 1$, after some algebraic manipulation and using Lemma 3 and Lemma 6, we get the desired result. □

Similarly, considering the matrix product

$$\begin{pmatrix} B_{k,x} & B_{k,x} \\ B_{k,z} & C_{k,z} \end{pmatrix} \begin{pmatrix} C_{k,y} \\ B_{k,y} \end{pmatrix} = \begin{pmatrix} B_{k,x+y} \\ B_{k,y+z} \end{pmatrix}$$

and proceeding in the same way as in the previous lemma, we get the following result.

Lemma 10. *For all integers x, y, and z, $B_{k,x+y}^2 - B_{k,x+y}B_{k,y+z}C_{k,z-x} - B_{k,y+z}^2 = B_{k,z-x}^2$, where $x \neq z$.*

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