

## A NOTE ON ZERO DIVISOR GRAPH WITH RESPECT TO ANNIHILATOR IDEALS OF A RING

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ABSTRACT. The zero divisor graph has been investigated in general for a commutative ring  $R$ . We consider a (not necessarily commutative) ring as a right module over itself. We consider annihilators of right ideals of  $R$  and define a graph related to these annihilators. Let  $R$  be a ring and  $I$  be an ideal of  $R$ . We denote the annihilator of  $I$  (viewed as a right  $R$ -module) by  $\text{Ann}(I)$ . We define a graph with respect to  $\text{Ann}(I)$  as follows and denote it by  $\Gamma_{A(I)}(R)$ :

$$\Gamma_{A(I)}(R) = \{E = (a, b) \mid a \in I, b \in \text{Ann}(I)\}.$$

With this we prove that for a right ideal of a ring  $R$  if  $I^* \cap \text{Ann}(I)^* = \phi$ , then  $\Gamma_{A(I)}(R)$  is bipartite, where  $K^* = K \setminus \{0\}$  for any subset  $K \subseteq R$ .

### 1. INTRODUCTION

The concept of zero divisor graph has been an active area of research since the notion was introduced by Beck [5]. Different aspects of zero divisor graph have been studied and investigations are on.

In most of the cases zero divisor graph of a commutative ring has been investigated.

Let  $R$  be a commutative ring with identity  $1 \neq 0$ . Let  $Z(R)$  be the set of zero divisors of  $R$  and  $Z(R)^* = Z(R) \setminus \{0\}$ . Two elements  $a, b \in Z(R)^*$  are adjacent if and only if  $ab = 0 = ba$ . The zero divisor graph of  $R$  is denoted by  $\Gamma(R)$ .

For any graph  $G = (V, E)$ , the set of vertices shall be denoted by  $V(G)$  and the set of edges shall be denoted by  $E(G)$ .

The zero divisor graph has been studied for the ring of continuous functions by Azarpanah and Motamedi [4], for a semiprime Gifand ring by Samei [10]. A relation of  $\Gamma(R)$  for a reduced ring has been given with respect to the prime radical of  $R$  (Samei [10, Theorem 3.1]).

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For more details and results the reader is referred to Anderson et al. [1, 2, 3]. Some more treatment could be found in Levy and Shepiro [6], Akbari et al. [1], Maimani et al. [7] and Samei [10].

## 2. ZERO DIVISOR GRAPH WITH RESPECT TO ANNIHILATORS

We continue the above investigation. We relate zero divisor graph of a (not necessarily commutative) ring to annihilator primes. Let  $R$  be a ring and  $I$  be a right ideal of  $R$ . Let  $A = \{a \in R \mid a \text{ is regular}\}$ ,  $B = \{b \in R \mid b \text{ is a zero divisor}\}$ , and  $B^* = B \setminus \{0\}$ . Furthermore,  $\text{Ann}(I) = \{r \in R \mid ar = 0, \text{ for all } a \in I\}$ . We know that  $\text{Ann}(I)$  is an ideal of  $R$ . Let  $\text{Ann}(I)^* = \text{Ann}(I) \setminus \{0\}$  and  $I^* = I \setminus \{0\}$ . We introduce zero divisor graph with respect to  $I$  in the following way. Let  $a \in I^*$ ,  $b \in B^*$ . Then  $a, b$  are adjacent if and only if  $ab = 0$ . We denote the graph by  $\Gamma_{A(I)}(R)$ . We note that  $\Gamma_{A(I)}(R)$  is a directed graph and  $V(\Gamma_{A(I)}(R)) \subseteq B^*$  and  $\text{Ann}(I)^* \subseteq B^*$ . If  $b \in \text{Ann}(I)^*$ , then obviously  $ab = 0$  for all  $a \in I^*$ . With this we prove the following:

**Theorem 1.** *Let  $R$  be a ring and  $I$  be a right ideal of  $R$ . If  $I^* \cap \text{Ann}(I)^* = \phi$  and we ignore isolated vertices, then  $\Gamma_{A(I)}(R)$  is bipartite.*

*Proof.*  $I^* \cap \text{Ann}(I)^* = \phi$  implies that if  $E = (a, b) \in \Gamma_{A(I)}(R)$ , then  $a \in I^*$ ,  $a \notin \text{Ann}(I)^*$  and  $b \in \text{Ann}(I)^*$ ,  $b \notin I^*$ .  $\square$

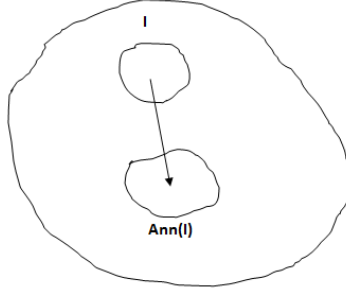


FIGURE 1

**Example 2.** Let  $R = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, b, c \in \mathbb{C} \right\}$  and  $I = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & c \end{pmatrix} \mid c \in \mathbb{C} \right\}$ .

Then  $I$  is a right ideal of  $R$  and  $\text{Ann}(I) = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \mid a, b \in \mathbb{C} \right\}$ . Now clearly  $I^* \cap \text{Ann}(I)^* = \phi$ , therefore,  $\Gamma_{A(I)}(R)$  is bipartite.

**Example 3.** Let  $R = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{Z} \right\}$  and  $J = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \mid a, b \in \mathbb{C} \right\}$ .

Then  $J$  is a right ideal of  $R$  and  $\text{Ann}(J) = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\}$ . We note that  $J$  is

faithful as a right  $R$ -module. (Recall that a right module  $M$  over a ring  $R$  is called faithful if  $\text{Ann}(M) = \{0\}$ ). We note that here  $\Gamma_{A(I)}(R)$  is the null graph.

Recall that  $\mathbb{Z}_n = \{0, 1, 2, \dots, n-1\}$  (where  $n$  is a positive integer) is a ring under addition modulo  $n$  and multiplication modulo  $n$ .

**Example 4.** Let  $R = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \mid a, b \in \mathbb{Z}_n \text{ where } n \text{ is some positive integer} \right\}$  and

$$J = \left\{ \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \mid a \in \mathbb{Z}_n \right\}.$$

Then  $J$  is a right ideal of  $R$  and we see that  $\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ , therefore,  $J^* \cap \text{Ann}(J)^* \neq \phi$  which implies that  $\Gamma_{A(J)}(R)$  is not bipartite. We also note that the zero divisor graph of  $J$  as a ring (i.e.  $\Gamma(J)$ ) is complete. Here for all  $a, b \in J$  we have  $ab = 0 = ba$ .

**Example 5.** Let  $R = M_2(\mathbb{Z}_2)$  and

$$I = \left\{ A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, C = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \right\}.$$

Then  $I$  is a right ideal of  $R$  and

$$\text{Ann}(I)^* = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\}.$$

Therefore,  $I$  is faithful as a right  $R$ -module and  $\Gamma_{A(I)}(R)$  is the null graph.

**Example 6.** Let

$$\begin{aligned} R &= \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, b, c \in \mathbb{Z}_2 \right\}, \\ I &= \left\{ \begin{pmatrix} 0 & 0 \\ 0 & c \end{pmatrix} \mid c \in \mathbb{Z}_2 \right\}, \\ I^* &= \left\{ A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}, \\ \text{Ann}(I)^* &= \left\{ B = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, C = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, D = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right\}, \\ AB &= \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \\ AC &= \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \\ AD &= \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

We see that  $\Gamma_{A(I)}(R) = K_{1,3}$  is complete bipartite.

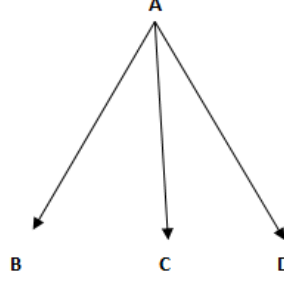


FIGURE 2

**Example 7.** Consider  $\mathbb{Z}_3 = \{0, 1, 2\}$ . Let  $R = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, b, c \in \mathbb{Z}_3 \right\}$  and  $I = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix} \mid a \in \mathbb{Z}_3 \right\}$ . Then  $I$  is a right ideal of  $R$ . Now

$$I^* = \left\{ A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, B = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} \right\}$$

and

$$\begin{aligned} P = \text{Ann}(I)^* \\ = \left\{ C = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, D = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, F = \begin{pmatrix} 2 & 2 \\ 0 & 0 \end{pmatrix}, \right. \\ \left. G = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}, H = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}, J = \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}, K = \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix} \right\}. \end{aligned}$$

Now  $P \cap I^* = \phi$ . Therefore,  $\Gamma_{A(I)}(R)$  is bipartite.

**Proposition 8.** Let  $P$  be defined as in Example 7. Then  $P \cup \{0\}$  is a prime ideal of  $R$ .

*Proof.* Since

$$ARB \in P = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \mid a, b \in \mathbb{Z}_3 \right\}$$

is valid,

$$\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} R \begin{pmatrix} u & v \\ 0 & u \end{pmatrix} \in \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \right\}.$$

Therefore,

$$\begin{pmatrix} aRu & arv + bRu \\ 0 & aRu \end{pmatrix} \in \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \right\}.$$

Thus,

$$aRu = 0$$

and hence

$$a = 0 \text{ or } u = 0.$$

If  $a = 0$ , we have  $A = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \in P$ .

If  $u = 0$ , we have  $B = \begin{pmatrix} 0 & v \\ 0 & 0 \end{pmatrix} \in P$ . Hence  $P$  is a prime ideal of  $R$ .  $\square$

### 3. ZERO DIVISOR GRAPH WITH RESPECT TO IDEALS

The concept of zero divisor graph with respect to an ideal was introduced by Redmond [9]. Let  $R$  be a commutative ring with identity  $1 \neq 0$  and  $I$  an ideal of  $R$ . The zero divisor graph with identity respect to  $I$  is denoted by  $\Gamma_I(R)$  and

$$\Gamma_I(R) = \{a \in R \setminus I \text{ such that } ab \in I \text{ for some } b \in R \setminus I\}$$

with distinct vertices  $a$  and  $b$  adjacent if and only if  $ab \in I$ . Thus if  $I = \{0\}$ , then  $\Gamma_I(R) = \Gamma(R)$ . Redmond [9] found a relationship between  $\Gamma_I(R)$  and  $\Gamma(R/I)$ , and proved that for a finite ideal of a commutative ring  $R$ ,  $\Gamma_I(R)$  contains  $|I|$  distinct subgraphs isomorphic to  $\Gamma(R/I)$ , where  $|I|$  denotes the order of  $I$ .

Maimani et al. [7, Theorem 2.2] have proved the following concerning isomorphisms of zero divisor graphs. Let  $R$  and  $S$  be two rings. Let  $I$  be a finite ideal of  $R$  and  $J$  be a finite ideal of  $S$  such that  $\sqrt{I} = I$  and  $\sqrt{J} = J$ . Then the following hold.

- (1) If  $|I| = |J|$  and  $\Gamma(R/I) \cong \Gamma(S/J)$ , then  $\Gamma_I(R) \cong \Gamma_J(S)$ .
- (2) If  $\Gamma_I(R) \cong \Gamma_J(S)$ , then  $\Gamma(R/I) \cong \Gamma(S/J)$ .

*Remark 9.* Let  $P$  be a prime ideal of a commutative ring  $R$ . Then  $\Gamma_P(R)$  is the null graph.

We take the notion of zero divisor graph with respect to an ideal of a commutative ring further in noncommutative set up in the following way.

**Definition 10.** Let  $R$  be a (not necessarily commutative) ring with identity  $1 \neq 0$  and  $I$  be a right ideal of  $R$ . The zero divisor graph with respect to  $I$  is denoted by  $\Gamma_I(I_R)$  and

$$\Gamma_I(I_R) = \{a \in R \setminus I \text{ such that } ab \in I, ba \in I \text{ for some } b \in R \setminus I\}$$

with distinct vertices  $a$  and  $b$  adjacent if and only if  $ab \in I$  and  $ba \in I$ .

**Example 11.** Let  $A = \mathbb{Z}_6 = \{0, 1, 2, 3, 4, 5\}$  and  $R = M_2(A)$ . Now  $I = \{0, 3\}$  is an ideal of  $A$  and  $K = M_2(I)$  is an ideal of  $R$ . Let  $U = \begin{pmatrix} 3 & 2 \\ 0 & 4 \end{pmatrix} \in R$ ,  $V = \begin{pmatrix} 1 & 3 \\ 0 & 0 \end{pmatrix} \in R$ , and  $W = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \in R$ . Then  $U \notin K$ ,  $V \notin K$  and

$W \notin K$ . Now we see that  $UV = \begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix} \in K$ ,  $VU = \begin{pmatrix} 3 & 2 \\ 0 & 0 \end{pmatrix} \notin K$ ,  $UW = \begin{pmatrix} 3 & 3 \\ 0 & 0 \end{pmatrix} \in K$ ,  $WU = \begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix} \in K$ . Therefore,  $U, W \in \Gamma_K(K_R)$ .

**Example 12.** Let  $S = \mathbb{Z}_2 = \{0, 1\}$  and  $R = M_2(S)$ . Now

$$I = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\}$$

is a right ideal of  $R$ . Also

$$\begin{aligned} R \setminus I &= \left\{ A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, C = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, D = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \right. \\ E &= \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, F = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, G = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, H = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \\ I &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, J = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, K = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, L = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \left. \right\}. \end{aligned}$$

We see that  $\Gamma_I(I_R) = \{F, D\}$  as  $FD \in I$  and  $DF \in I$ .

**Remark 13.** Let  $P$  be a completely prime ideal of a ring  $R$ . Then  $\Gamma_P(P_R)$  is the null graph.

Recall that an ideal  $P$  of a ring  $R$  is completely prime if  $R/P$  is a domain, i.e.,  $ab \in P$  implies  $a \in P$  or  $b \in P$  for  $a, b \in R$  (McCoy [8]).

**Remark 14.** Let  $R$  be a (not necessarily commutative) ring with identity  $1 \neq 0$  and  $I$  be a right ideal of  $R$ . Then  $\Gamma_I(I_R) \cup \{0\}$  need not be a right ideal of  $R$ .

We saw in Example 12 that  $D + F = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \notin \Gamma_I(I_R)$ .

We note that in Example 11 there were vertices  $U \notin I$ ,  $V \notin I$  such that  $UV \in I$  but  $VU \notin I$ . Similarly in Example 12 we had  $AB \in I$ ,  $BA \notin I$ ,  $AD \in I$ ,  $DA \notin I$ ,  $CA \in I$ ,  $AC \notin I, \dots, LE \in I$ ,  $EL \notin I$ ,  $LF \in I$ ,  $FL \notin I$ .

This motivates one to define the graph (directed) with respect to a right ideal in the following way. Here we use the notation as in Redmond [9].

**Definition 15.** Let  $R$  be a (not necessarily commutative) ring with identity  $1 \neq 0$  and  $I$  be a right ideal of  $R$ . The zero divisor graph (directed) with respect to  $I$  is denoted by  $\Gamma_I(R)$  and is defined as

$$\Gamma_I(R) = \{a \in R \setminus I \text{ such that } ab \in I \text{ for some } b \in R \setminus I\}$$

with distinct vertices  $a$  and  $b$  adjacent if and only if  $ab \in I$ .

**Remark 16.** Let  $R$  and  $I$  be as in Example 12. Then

$$\Gamma_I(R) = \{A, C, D, F, H, J, K, L\}.$$

We note that  $B \notin \Gamma_I(R)$  as there does not exist any element  $T \in R \setminus I$  such that  $BT \in I$ .

*Remark 17.* Let  $R$  be a (not necessarily commutative) ring with identity  $1 \neq 0$  and  $I$  be a right ideal of  $R$ . Then  $\Gamma_I(R) \cup \{0\}$  need not be a right ideal of  $R$ .

We saw in Example 12 that  $A + C = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \notin \Gamma_I(R)$ .

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